Anomalous Phase Synchronization in Coupled Chua's oscillator

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Abstract: We report the first experimental evidence of anomalous phase synchronization in two coupled Chua's oscillators. The anomalous onset of phase synchronization shows an increasing frequency disorder with coupling strength before following a usual transition to phase synchronization as monotonic decrease in frequency disorder with a maximum at an intermediate coupling. The effect of coupling asymmetry on anomalous phase synchronization is also revealed.

Index Terms: Chua's oscillator, Anomalous phase synchronization, antiphase synchronization, frequency disorder.

I. INTRODUCTION

The studies on synchronization of interacting nonlinear oscillators are of fundamental importance in many areas ranging from physical to living systems [1-2]. The theories of synchronization have been extended [3] to chaotic systems in 1990 with unidirectional coupling of two identical oscillators. Both the amplitudes and phases of the coupled systems are strongly correlated in such complete synchronization (CS). On the other hand, phase synchronization (PS) [1, 2, 4] has been found to play crucial roles in many weakly interacting living systems, and also in laboratory experiments [5]. PS is observed in physical experiment [6], cardio-respiratory rhythm [7], neural oscillator [8], and behavioral psychology [9] and in ecology [10]. In case of PS, two or many interacting oscillators with varying natural frequencies develop a phase locking relation \( m:n \) (\( m \) and \( n \) are integers) for weak coupling although the amplitudes remain almost uncorrelated. The coupled oscillators rotate with a common frequency at weaker coupling above a critical value.

Similar systems show variations in the frequency of oscillations due to natural parameter mismatch, which is defined as frequency disorder. Coupling suppresses this frequency disorder of the interacting oscillators to a phase locking state. The common notion [2, 11] of transition to PS is that the frequency disorder decreases monotonically with coupling and disappears above a critical value. Recently, a mark departure from this common notion has been observed [12-13] in a Foodweb model [10]. A large population of nonidentical oscillators shows an initial increase in frequency disorder followed by monotonic decrease with coupling. Such an unusual enlargement of frequency disorder with an intermediate maximum is denoted as anomalous phase synchronization (APS). APS is established both theoretically and analytically as a universal phenomenon, which may be observed in any coupled system and even only in two interacting systems. The exact condition for APS in any system can be derived with an appropriate choice of system parameters when two or more parameters of a system are functionally dependent. APS originates in the nonisochronicity of oscillations and arises when nonisochronicity increases with the natural frequency of oscillation. Nonisochronicity as a shear of phase flow [14] induces an amplitude dependence of natural frequency of an oscillator. Synchronization can also be enhanced, in contrast, by this mechanism when nonisochronicity and natural frequency have negative covariance. This opens the door for strategies of synchronization control. With an appropriate choice of system’s parameters, the effect of APS may be induced either for enhancing or inhibiting synchronization. Similar strategies can be adopted in biological systems and thus APS can play an important role in living systems too.

In this paper we report the first experimental evidence of APS in two coupled nonidentical Chua's oscillators. Two diffusively coupled Chua's oscillators show the anomalous transition to in-phase synchronization for comparatively large frequency mismatch while usual transition to antiphase is observed for smaller frequency mismatch at much weaker coupling. The effect of coupling asymmetry is also very clear from this experiment. On reversing the sign of frequency mismatch, PS is enhanced to antiphase at a very weak coupling. However, we still find anomalous transition to PS in the antiphase regime for different set of system parameters. The antiphase regime includes out-of-phase states when phase difference of interacting systems remains bounded to \( 0< \Delta \phi < \pi \). We denote both the antiphase \(( \Delta \phi = \pi \)\) and the out-of-phase \(( 0< \Delta \phi < \pi \)\) states as antiphase regime since they remain in a synchronization manifold transverse to the in-phase synchronization manifold. In case of APS, smooth variations in the attractor topology are only observed due to the shear of phase flow, which enables the enlargement of frequency disorder in the weak coupling regime. We have also observed anomalous transition to both
in-phase and antiphase in the extreme case of coupling asymmetry for unidirectional coupling.

II. COUPLED CHUA'S CIRCUIT

The Chua's circuit is one of the simplest electronic oscillators, which shows limit cycle to chaotic oscillations through different routes as period-doubling, period-adding bifurcation or torus breakdown [15]. Two coupled nonidentical Chua's oscillators are shown in Fig.1, where each oscillator consists of linear passive elements as resistor \( R_{1,8} \), inductor \( L_{1,2} \), capacitors \( C_{1,3} \), \( C_{2,4} \) and one nonlinear resistance. The nonlinear resistance is approximated by a piecewise linear function and designed by using a pair of linear amplifiers (U1-U2 or U3-U4;µA741) for each oscillator. A unity gain amplifier (U5;µA741) with a series resistance \( R_c \) is used for unidirectional coupling. The resistance \( R_c \) decides the coupling strength. The coupling strength increases with decrease in coupling resistance.

![Fig 1. Two coupled Chua’s oscillators: Power supply: ±9V.](image)

The state variables are the voltages \( V_{C1,C3} \) and \( V_{C2,C4} \) at respective capacitor nodes, and the inductor current \( I_{1,2} \). The natural frequency \( \omega_{2} \) of an uncoupled Chua’s oscillator is approximated [16] by

\[
\omega_{2} = \frac{1}{L_{2,8}C_{2,4}} \sqrt{1 + \frac{M_j}{R_{1,8}C_{1,3}}} \quad \text{where} \quad M_j = a_{1,2}h_{1,2}
\]

where \( a_{1,2} \), \( b_{1,2} \) are given by

\[
a_{1,2} = (-\frac{1}{R_{2,9}})R_{1,8}, \quad b_{1,2} = (\frac{1}{R_{3,10}} - \frac{1}{R_{5,12}})R_{1,8}
\]

Equation (1) clearly shows that the natural frequency of the Chua's oscillator is dependent on more than two parameters including the nonlinearity of the system, which is represented by the slopes \( (a_{1,2}, b_{1,2}) \) of the piecewise linear function. All the circuit components are different, since no two similar off-the-shelf components are found identical, which introduce the mismatch in the natural frequency of oscillations. All components are kept fixed throughout this paper except the resistance \( R_{1,8} \), which is varied to obtain different dynamical regimes, periodic and chaotic ones. Two similar state variables \( V_{C1} \) and \( V_{C2} \) from the two oscillators at capacitor nodes \( C_1 \) and \( C_2 \) respectively are monitored using a 2-channel digital oscilloscope (HP TDS 220) for varying coupling resistance \( R_c \). The instantaneous phases \( \phi_{2}(t) \) of measured signals are determined using the Hilbert transform [1-2] and the mean frequencies \( \Omega_{1,2} \) of the coupled oscillators are then estimated as the mean rate of change of \( \phi_{2}(t) \). A simple index of relative phase difference in terms of mean frequencies, \( \Delta \Omega(\epsilon) = 2(\Omega_{1}(\epsilon) - \Omega_{2}(\epsilon))/\Omega_{1}(\epsilon) + \Omega_{2}(\epsilon) \), is taken as a measure [10] of synchronization, which is the frequency difference as percentage of mean frequencies of the oscillators \( \Omega_{i}(\epsilon) \) \((i=1,2)\). Phase synchronization is established when the relative phase difference disappears \( (\Delta \Omega=0) \).

III. EXPERIMENT ON ANOMALOUS PHASE SYNCHRONIZATION

We controlled the natural frequency of the oscillators (uncoupled state) either to enhance or inhibit synchronization by appropriate choice of circuit parameters. The natural frequency of Chua’s oscillator depends upon more than one system parameter and the nonlinearity parameter. This functional dependence of two or more system parameters is a necessary criterion to observe APS.

For diffusive coupling (U5 is withdrawn), \( R_1 \) and \( R_8 \) are so selected that both the oscillators (OS-1, OS-2) are kept in limit cycle state and the natural frequency \( \Omega_{2}(\epsilon=0)=\omega_{0} \) of OS-1 is larger than \( \Omega_{2}(\epsilon=0)=\omega_{2} \), the frequency of OS-2. The frequency disorder \( \Delta \Omega(\epsilon) \) in Fig.2a shows increasing trend at weak coupling and then decreases with coupling before disappearing \( (\Delta \Omega=0) \) above a critical coupling. Evidently, the frequency disorder starts growing without any threshold and the traces (solid and dotted lines) for two different frequency mismatch, \( \Delta \Omega=\omega_{0} - \omega_{0}=\Delta \Omega(0) \), show a shift in critical coupling with mismatch. The coupling threshold increases with mismatch.

A sufficient condition for APS is defined [13] as \( d\delta/d\omega>0 \) where \( \kappa \) is the slope of individual frequency with coupling. This condition is clearly satisfied in our experiment as evident from the plot of individual frequency \( \Omega_{2}(\epsilon) \) with coupling in Fig.2c. The slope of \( \Omega_{1} \) (solid line) as defined as \( d\Omega_{2}/d\epsilon \) is larger than \( d\Omega_{2}/d\epsilon \) (dotted line) as observed during the increasing trend of \( \Delta \Omega(\epsilon) \) in the weaker coupling \( (2.5E-6<\epsilon<1E-5) \) range. The larger the individual frequency, the higher is the slope. This confirms the existence of APS. It may be noted here that, in uncoupled state, the individual frequency \( \Omega_{2}(0)=\Omega_{2}(0) \). The situation changes dramatically with reduced frequency mismatch. The synchronization is enhanced by a monotonic decrease in
frequency disorder $\Delta \Omega(\epsilon)$ with transition to antiphase above a coupling threshold, the details of which is not presented here due to short of space. We rather concentrated on APS.

![Anomalous phase synchronization](image)

**Fig.2.** Anomalous phase synchronization: OS-1, OS-2 are both period-1 oscillators, (a) $\Delta \Omega(\epsilon)$ with coupling for $\Delta \omega = 279\, \text{Hz}$ (solid trace, $R = 1570\, \Omega$, $R_c = 1449\, \Omega$) and $289\, \text{Hz}$ (dotted trace, $R = 1570\, \Omega$, $R_c = 1447\, \Omega$), (b) individual frequencies $\Omega_{1,2}(\epsilon)$ with coupling for $\Delta \omega = 279\, \text{Hz}$, (c) magnified version of (b) for $2.5E-6 < \epsilon < 1E-5$.

So far anomalous transition to in-phase is reported here as experimental evidences of earlier works on APS [12-13]. In addition, anomalous transition to antiphase is first observed in our experiment by reversing the natural frequency mismatch when $\Omega_{1}(0) < \Omega_{2}(0)$. Antiphase synchronization has been reported earlier in neural system [17] and in recurrent epidemic [18], but no anomalous transition to PS was ever reported. Also for $\Delta \omega = 124\, \text{Hz}$ (dotted line) we observed enhanced transition to antiphase as shown in Fig.3. For a larger mismatch $\Delta \omega = 180\, \text{Hz}$ (dashed line), almost usual transition to antiphase is observed. But for further increase in $\Delta \omega = 219\, \text{Hz}$ (solid line), APS is clearly evident. The out-of-phase and antiphase states as elaborated above are also observed in the antiphase regime. Large desynchronization with fluctuations in frequency disorder is also found in this case, during the transition from antiphase to in-phase for intermediate coupling. A coupling threshold is seen for the start of desynchronization, however, no decisive statement can be made, at this point, regarding the nature of shift in this coupling threshold, which requires further investigations with rigor. We restrict our report, in this paper, to the anomalous transition to PS only. During transition from desynchronization to in-phase shown in Fig.3(a), the coupling threshold again increases with mismatch.

![APS in antiphase regime](image)

**Fig.3.** APS in antiphase regime: $\Delta \Omega(\epsilon)$ increases with coupling for $\Delta \omega = 219\, \text{Hz}$ (solid line) with $R_c = 1452$ (OS-1: chaotic), $R = 1540\, \Omega$ (OS-2: period-1), and for $\Delta \omega = 180\, \text{Hz}$ (dashed line) with $R_c = 1445\, \Omega$ (OS-1: chaotic), $R = 1530\, \Omega$ (OS-2: period-1), $\Delta \omega = 124\, \text{Hz}$ (dotted line) with $R_c = 1468\, \Omega$ (OS-1: chaotic), $R = 1516\, \Omega$ (OS-2: period-1), (b) magnified version of (a) in the coupling range $2.5E-6 < \epsilon < 1E-5$, (c) individual frequencies $\Omega_{1,2}$ for $\Delta \omega = 219\, \text{Hz}$ in the coupling range $3.5E-6 < \epsilon < 1E-5$.

The nature of transition to PS depends upon the sign of natural frequency mismatch, i.e., whether $\Delta \omega$ is positive or negative. The sign of mismatch is arbitrary for diffusive coupling, nevertheless we can change from anomalous to usual or enhanced transition to PS by reversing the natural frequency mismatch ($\Delta \omega$) and thereby able to control synchronization either to enhance or to inhibit it. It reflects the effect of coupling asymmetry.

The effect of coupling asymmetry is clearly understood in the extreme case of asymmetry for unidirectional coupling. For unidirectional coupling the linear amplifier U5 is connected. The driver oscillator OS-1 is made chaotic and the response OS-2 is kept at limit cycle oscillation (period-2)
by appropriate selections of $R_1$ and $R_2$ respectively, when the natural frequency of response (OS-2) oscillator ($\omega_2$) is larger than the driver oscillator (OS-1) frequency ($\omega_1$). Anomalous transition to in-phase synchronization (solid line) is seen in Fig.4(a) for selected parameters with frequency mismatch ($\Delta \omega=127\text{Hz}$). In a reverse situation, when the response frequency is lower than the driver frequency, enhanced transition to antiphase synchronization (dotted line) is observed for very weak coupling. All the synchronization states, namely, out-of-phase, antiphase are also observed before the onset of large desynchronization for intermediate coupling. But the transition to in-phase synchronization after large desynchronization occurs for much stronger coupling as compared to anomalous transition to in-phase.

![Fig.4. APS with unidirectional coupling: (a) OS-1 chaotic driver (2533Hz) and OS-2 period-2 response (2660Hz), $\Delta \omega=127\text{Hz}$, APS in solid line. For periodic driver (2563Hz) and chaotic response (2458Hz), usual transitions to antiphase and then to in-phase are found (dotted line) for $\Delta \omega=105\text{Hz}$, $R_1=1446\Omega$, $R_2=1484\Omega$, (b) magnified versions of (a) in the range 2.5E-6 to 1.5E-5, (c) $\Omega_{12}$ for $\Delta \omega=127\text{Hz}$. The response frequency (solid line) is enlarged while the driver frequency remains unaffected (dotted line). A part of large desynchronization (dotted line) is seen in (b) for $\epsilon=1.5E-5$.](image-url)

IV. CONCLUSION

The first experimental evidence of anomalous transition to PS is reported using two coupled nonidentical Chua's oscillators. Anomalous transition to antiphase is first observed in our experiment. Desynchronization with large fluctuations in frequency disorder has been observed for intermediate coupling in a turbulent phase of coexisting in-phase and antiphase states during the transition from antiphase to in-phase. Although a number of recent studies reported such phenomenon in many systems, a clear understanding of it is yet to be arrived at.

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REFERENCES