

Transition to synchronization in weakly coupled oscillators with nearest neighbour coupling

H.F. El-Nashar,^{1,2} P. Muruganandam,³ F.F. Ferreira,⁴ and H.A. Cerdeira^{5,6}

¹*Department of Physics, Faculty of Education, Riyadh University for Girls, Alkharj 11942, P.O.Box 21034, K.S.A.*

²*Department of Physics, Faculty of Science, Ain Shams University, Cairo 11566, Egypt.*

³*School of Physics, Bharathidasan University, Tiruchirapalli 620 024, India.*

⁴*Grupo Interdisciplinar de Física da Informação e Economia (GRIFE),
Escola de Arte, Ciências e Humanidades, Universidade de São Paulo,
Av. Arlindo Bettio 1000, 03828-000 São Paulo, Brazil.*

⁵*Instituto de Física Teórica, Universidade Estadual Paulista, R. Pamplona 145, 01405-900 São Paulo, Brazil.*

⁶*Instituto de Física, Universidade de São Paulo, R. do Matão, Travessa R. 187, 05508-090 São Paulo, Brazil.*

We study a model of weakly coupled oscillators with nearest neighbours coupling. We show that a condition of a stable phase locked solution is decided by the oscillators at the borders between the major clusters, which merge to form a larger one of all oscillators at the stage of complete synchronization. Therefore, we are able to get an analytic form for the phases, frequencies as well as the periodic time of oscillation. In addition, we get a condition for phase synchronization to be observed.

I. INTRODUCTION

Systems of coupled oscillators can describe problems in physics, chemistry, biology, neuroscience and other disciplines. They have been used to model several phenomena such as: Josephson junction arrays, multimode lasers, vortex dynamics in fluids, biological information processes, neurodynamics [1–3]. These systems have been observed to synchronize themselves to a common frequency, in a tree-like behaviour, when the coupling strength between these oscillators is increased [4–12]. The synchronization features of many of the above mentioned in spite of the diversity of the dynamics, might be described using simple model of weakly coupled phase oscillators such as Kuramoto model [8].

Although, there has been an extensive exploration of the dynamical behaviour of the Kuramoto model (global coupling among all oscillators), little has been done regarding the local model of nearest neighbour interactions which can be considered as a diffusive version of the Kuramoto model, which we shall call local Kuramoto model in this paper. This model has been used recently to describe Josephson Junction arrays in a ladder [13] and locally coupled lasers [14]. This one dimensional model can be expressed as [15–18]:

$$\dot{\theta}_i = \omega_i + \frac{K}{3} [\sin(\theta_{i+1} - \theta_i) + \sin(\theta_{i-1} - \theta_i)], \quad (1)$$

where ω_i are the natural frequencies, K is the coupling strength, θ_i is the instantaneous phase, $\dot{\theta}_i$ is the instantaneous frequency and $i = 1, 2, \dots, N$. Many interesting features of the locally coupled Kuramoto model remain unknown, especially an analytic solution [14].

According to (1), the nonidentical oscillators cluster in time averaged frequency, until they completely synchronize to a common value of the average frequency at a critical coupling K_c [15–19]. When the time averaged frequency is considered, they start forming clusters much earlier than K_c . At $K \geq K_c$, and for periodic boundary conditions $\theta_{i+N} = \theta_i$, complete frequency synchronization can be observed to a common frequency value $\omega_0 = \frac{1}{N} \sum_{i=1}^N \omega_i$, where phases and frequencies are time independent. In Fig. 1 we show the synchronization tree for a periodic system with $N = 15$ and $N = 20$

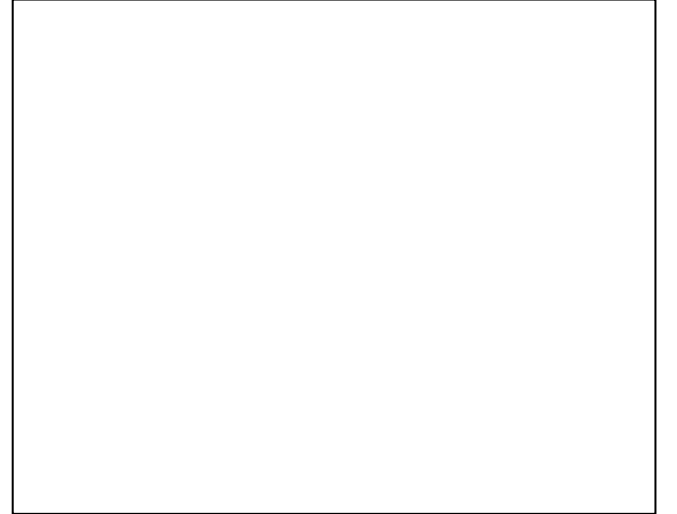


FIG. 1: Synchronization tree for a system of (a) $N = 15$ oscillators and (b) $N = 20$ oscillators.

oscillators, where the elements which compose each one of the major clusters that merge into one at K_c , are indicated in each branch.

II. PHASE-LOCKED SOLUTION

In terms of phase differences $\phi_i = \theta_{i+1} - \theta_i$, system (1), can be rewritten as:

$$\dot{\phi}_i = \omega_{i+1} - \omega_i + \frac{K}{3} [\sin \phi_{i-1} - 2 \sin \phi_i + \sin \phi_{i+1}], \quad (2)$$

with $\dot{\phi}_i^* = 0$ at K_c for $i = 1, 2, \dots, N$. In addition, all quantities (phase-difference ϕ_i^* and frequency $\dot{\theta}_i^* = \omega_0$) become time independent [15–18] at the critical coupling. Attempts to get a solution of the above equation (2) [20, 21] show that for only one value of l among the set $1, 2, \dots, N$, results $|\sin \phi_l^*| = 1$ and this is a necessary condition for equation

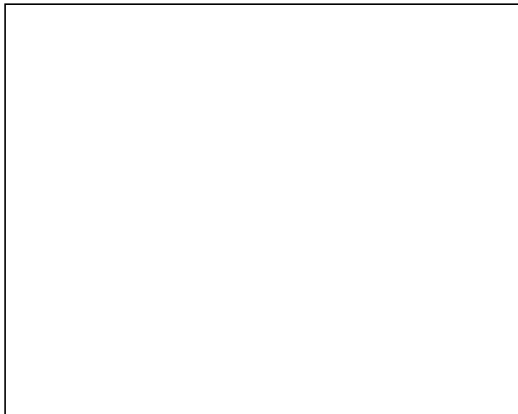


FIG. 2: Values of $\sin \phi_i$ for (a) $N = 15$ and (b) $N = 20$.

(2) to have a phase-locked solution. This fact has been used in reference [13] to estimate the value of K_c , at which synchronization occurs. However, the determination of which is the value of l , for which $|\sin \phi_l^*| = 1$, remains difficult.

It has been found numerically that, at the onset of synchronization, the values of $\dot{\theta}_i(t)$ and $\dot{\phi}_i(t)$ remain equal to ω_0 and zero, respectively, for a time T and then they burst [15, 19]. During this time period T , in between bursts, the quantity $\phi_i(t)$ remains fixed and it has an abrupt change by an amount 2π at each burst (phase-slip behaviour). The periodic time T blows up as K becomes close to K_c and $T = \infty$ at K_c . A detailed study of the temporal behaviour of all quantities of $\sin \phi_i^*$ for several values of N and for different sets of ω_i , shows that only one value of $|\sin \phi_l^*| = 1$. Figure (2) shows clearly such observations for $N = 15$ and for $N = 20$. We find that the value of $|\sin \phi_l^*| = 1$ holds for only one value of l where the two oscillators $l + 1$ and l belong to different clusters. It has been found also that these two oscillators previously mentioned are at the borders between the major clusters which merge at K_c .

In fact the burst and phase slip of quantities $\dot{\phi}_i(t)$ and $\phi_i(t)$, respectively, as well as the foundation of $|\sin \phi_l^*| = 1$, will allow us to rewrite (2) as:

$$\dot{\phi}_l^* = A - 2 \sin \phi_l^*, \quad (3)$$

where, $A = \frac{3|\omega_{l+1} - \omega_l|}{K} + \sin \phi_{l-1}^* + \sin \phi_{l+1}^*$ and at K_c , $\phi_l^* = \pi/2$ and, ϕ_{l-1}^* and ϕ_{l+1}^* are constants and time-independent as well as $A = 2$. Equation (3) is similar to that of a nonuniform oscillator, where a saddle-node bifurcation occurs at K_c . Therefore, in the vicinity of K_c from below, the value of ϕ_l arrives to zero as ϕ_l goes to $\pi/2$, in the phase space. It feels a half stable fixed point and remains for a time T and then suddenly escapes away. Correspondingly, $\dot{\phi}_l$ and ϕ_l have a burst and phase-slip, respectively. Thus, T increases as K goes to K_c and it diverges when the system arrives at the fixed point. Fig. 3 shows this feature for $\dot{\phi}_l$ versus ϕ_l for $N = 15$ close to K_c . The difference between the theoretical nonuniform oscillator and the measured one is due to the time variation of quantities $\phi_{l-1}(t)$ and $\phi_{l+1}(t)$ before K_c .

At K_c , Table I shows clearly the fact that $A \sim 2$ for several

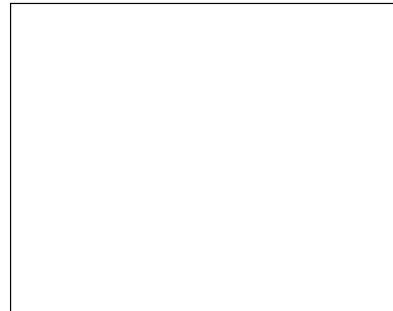


FIG. 3: Behaviour in the phase space of $\dot{\phi}_{13}$ in the vicinity of K_c for $N = 15$ oscillators (numerical) in comparison to the nonuniform oscillator (theoretical).

TABLE I: Measured values of K_c and A for different values of N .

N	K_c	A
3	0.850412	1.992
5	3.170827	2.002
10	3.547010	1.994
15	3.870510	1.996
20	4.958300	2.003
25	3.641060	1.995
50	9.457200	1.991
100	12.723200	1.992

values of N . This A value has been checked for several values of N and for different values of ω_i for each N , and it is found that $A \sim 2$ at the critical coupling. Therefore, we argue



FIG. 4: The behaviour of $|\sin \phi_l|$ taken from equation (4) is plotted as a function of (a) A for a fixed value of time and (b) time for a fixed A at the vicinity of K_c .

that, the determination of l among the set of $i = 1, 2, \dots, N$, where $|\sin \phi_l^*| = 1$, allows us to arrive to eq. (3), which can



FIG. 5: The behaviour of $|\dot{\phi}_l|$ is plotted (a) versus A for a given value of time and (b) versus time for a fixed value of A at the vicinity of K_c .

be solved analytically and the solution is

$$\phi_l = 2 \tan^{-1} \left(\frac{\sqrt{A^2 - 4} \tan \left(\frac{1}{2} t \sqrt{A^2 - 4} \right) \pm 2}{A} \right), \quad (4)$$

$$\dot{\phi}_l = \frac{(A^2 - 4) \sec \left(\frac{1}{2} t \sqrt{A^2 - 4} \right)^2}{A \left[1 + \frac{(\pm 2 + \sqrt{A^2 - 4} \tan \left(\frac{1}{2} t \sqrt{A^2 - 4} \right))^2}{A^2} \right]}. \quad (5)$$

Equations (4) and (5) show that, at $A = 2$ (at K_c), the values $\sin \phi_l^* = \pm 1$ which leads to $\dot{\phi}_l^* = 0$. Also, it can be seen that for $A > 2$, the values of ϕ_l and $\dot{\phi}_l$ are time dependent. In addition, at the vicinity of K_c and below, $\sin \phi_l$ and $\dot{\phi}_l$ are time dependent, they remain constants for a period T and they have sudden bursts. In Fig. 4(a) we can see the behaviour of $|\sin \phi_l|$ as a function of A for a specific value of t , while in Fig. 4(b) we see its variation with time when $A \sim 2$ (close to K_c): $T = \pi\sqrt{2}/\sqrt{A(A-2)}$. According to Fig. 4, there

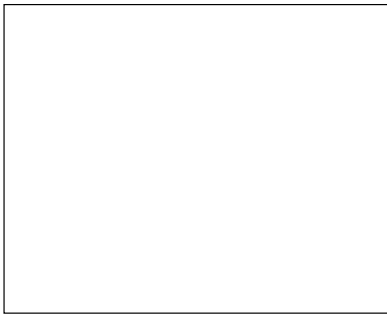


FIG. 6: $\log T$ versus A shows the divergence of the period T when A approaches 2.

is no time dependency for $|\sin \phi_l|$ when $A \leq 2$ and close to K_c , $|\sin \phi_l| = 1$ for T and there is no phase-slip for the quantity $|\phi_l|$ for $A \leq 2$, while this quantity has a phase-slip in between period T , when $A > 2$. Figure 5 shows the behaviour for $|\dot{\phi}_l|$. We can see the bursting behaviour that we have described above.

The periodic time T can be found analytically to be In Fig. 6 we clearly see that T blows up as A becomes close to 2 (where K goes to K_c), for the case of $N = 15$. According to Fig. 6, it is found that T blows up as $(A - 2)^{-0.5}$ which shows that a saddle node bifurcation occurs at K_c . Assuming that $\sin \phi_{l-1}$ and $\sin \phi_{l+1}$ remain constant for a time period T , in the vicinity of K_c , and equal to their values at K_c (it has been found numerically, that a very small deviation occurs which can be ignored), we find that: $T \sim \text{const.}/\sqrt{K_c - \frac{A}{2}K}$, where the value of $A \sim 2$ in the vicinity of K_c and $A = 2$ at K_c . So that, within a good approximation, the periodic time T blows up as $(K_c - K)^{-0.5}$, in good agreement with the numerical calculation by Zheng et. al [15], showing that a saddle-node bifurcation occurs at K_c .

According to equation (4) and the evidence of the saddle-node bifurcation which occurs at K_c , one can arrive to the following map [22] $x(t+1) = x(t) + \frac{A}{2}Tx^2(t) - \epsilon$, where $\epsilon = T(A-2)$ and $\frac{A}{2} \tan \frac{\phi_l(t)}{2} - 1 = x(t) - \epsilon$ which implies a saddle-node bifurcation at $A = 2$; i.e. at K_c [23].

III. THE MECHANISM OF COMPLETE SYNCHRONIZATION

In order to understand the mechanism of full synchronization which occurs at K_c , we use the fact that $\sin \phi_l^* = \pm 1$, where each $\dot{\theta}_i = \omega_0$. Hence, from system (1), we are able to get the following relations:

$$\sin \phi_{l+m}^* = \frac{3}{K_c} \sum_{m=1}^{N-l} (\omega_0 - \omega_{l+m}) \pm 1, \quad (6a)$$

$$\sin \phi_{l-n}^* = \frac{-3}{K_c} \sum_{n=1}^{l-1} (\omega_0 - \omega_{l-n}) \pm 1. \quad (6b)$$

Using this fact, we are able to write the following equations (where in the vicinity of K_c , for a time T , $\sin \phi_{l+m}$ and $\sin \phi_{l-n}$ remain unchanged in comparison to their values at exact K_c):

$$\phi_{l-n} = \sin^{-1}(a \pm \sin \phi_l), \quad \dot{\phi}_{l-n} = \frac{\cos \phi_l \dot{\phi}_l}{\sqrt{1 - (a \pm \sin \phi_l)^2}}, \quad (7)$$

$$\phi_l = 2 \tan^{-1} \left(\frac{\sqrt{A^2 - 4} \tan \left(\frac{1}{2} t \sqrt{A^2 - 4} \right) \pm 2}{A} \right), \quad (8)$$

$$\dot{\phi}_l = \frac{(A^2 - 4) \sec \left(\frac{1}{2} t \sqrt{A^2 - 4} \right)^2}{A \left[1 + \frac{(\pm 2 + \sqrt{A^2 - 4} \tan \left(\frac{1}{2} t \sqrt{A^2 - 4} \right))^2}{A^2} \right]}, \quad (9)$$

$$\phi_{l+m} = \sin^{-1}(b \pm \sin \phi_l), \quad \dot{\phi}_{l+m} = \frac{\cos \phi_l \dot{\phi}_l}{\sqrt{1 - (b \pm \sin \phi_l)^2}}, \quad (10)$$

where, $a = \frac{-3}{K_c} \sum_{n=1}^{l-1} (\omega_0 - \omega_{l-n})$ and $b = \frac{3}{K_c} \sum_{m=1}^{N-l} (\omega_0 - \omega_{l+m})$. It is clearly seen that according to the above equation,

each ϕ_i can be expressed in terms of ϕ_l and consequently each $\dot{\phi}_i$ can be expressed in terms of $\dot{\phi}_l$. Therefore, all values of ϕ_i will be shifted from each other by some certain constant which is determined according to the location of the index of the oscillator relative to oscillators with indexes l and $l + 1$. This is clearly shown from Fig. (2) where $\sin \phi_i$ values are shifted from each others, at K_c and no two values of are identical. This means that the burst behaviour starts first for $\dot{\phi}_l$ in

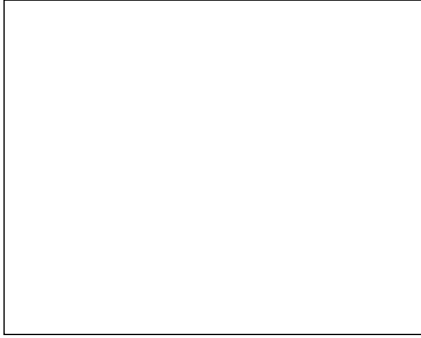


FIG. 7: Bursting behaviour of the time evolution of the phase differences between several oscillators, showing that their peaks shift from each other, close to K_c .

the vicinity of K_c due to the saddle-node bifurcation and then spread out through the ring in terms of the diffusive interaction between elements. Therefore, according to the equations above, all values of $\dot{\phi}_i$ will be shifted from the value of $\dot{\phi}_l$ and

this shift depends on the values a and b . Figure (7) shows $\dot{\phi}_l$ and some of the values of $\dot{\phi}_i$ where the shifts are presented. Henceforward, we argue that it is the behaviour of ϕ_l and $\dot{\phi}_l$, which drives the system to fall into full synchronization.

IV. CONCLUSION

In conclusion, we are able to determine a condition for which the full synchronization occurs at the critical coupling between oscillators for a local Kuramoto model. Such condition, which is $|\sin \phi_l^*| = 1$, allows us to solve one of the equations of system (2) analytically. Also, it has been found that a full synchronization occurs always when the variable $A = 2$ at K_c . Due to the diffusive nature of the local Kuramoto model, complete synchronization of all oscillators to a common value can be interpreted and understood once we have an analytic forms for ϕ_l^* and $\dot{\phi}_l^*$.

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