Abstract: Response of a hard, Duffing oscillator to harmonic excitation is investigated analytically, numerically and experimentally. A single degree-of-freedom torsional oscillator satisfying hard Duffing’s equation and exhibiting strong nonlinearity is designed. The system is experimentally investigated for possible complex and chaotic motions predicted by analytical and numerical treatments. The existence of multiple solutions, jump phenomenon, complex motion at low frequency, superharmonic and subharmonic response, chaotic motion and Wada basins of attraction are discussed.

Index Terms: Duffing Oscillator, Chaotic Response, Wada Basins.

1. INTRODUCTION

Duffing oscillator with cubic nonlinearity in the restoring force constitutes one of the important paradigms for studying dissipative mechanical oscillators exhibiting chaotic dynamics [1-3]. Soft oscillators with multiple equilibrium points, in comparison to hard oscillators with a single equilibrium point, in the unforced state have received wider attention in the literature. This is true for analytical, numerical and experimental investigations. In this paper, we consider a hard, damped Duffing oscillator under harmonic excitation. The equation of motion, in this situation can be written as,

\[ \ddot{x} + c\dot{x} + \alpha x + \beta x^3 = F \cos \omega t \]  

with \( c > 0, \alpha > 0 \) and \( \beta > 0 \).

The analytical treatment can be carried out by either perturbative or averaging methods [4-5]. The perturbative methods are applicable for systems with only weak nonlinearity, whereas the averaging methods do not suffer from this drawback. However, the first type of methods keeps track of the order of the error, in contrast to the latter, which does not give any estimate of the error. Up to the first order, all methods give the same result. In this paper we discuss only some general characteristics of the response as revealed by the first order approximation using the harmonic balance method. The stability analysis, superharmonic and subharmonic responses are included.

For experimental investigations, a torsional single degree-of-freedom oscillator, exhibiting hard cubic nonlinearity in the restoring force was designed following the idea proposed by Pippard [6]. The linearized natural frequency of the system turned out to be 5.7 Hz. The damping is modeled as linear viscous type and measured from the free vibration decay rate. The frequency response and the power spectrum were measured for various levels of excitation up to 35 Hz.

Using the experimentally obtained system parameters, the equation of motion is numerically integrated using a standard solver ode 45 in MATLAB 6.5 with different initial conditions. Interesting features suggested by analytical and/or experimental results were confirmed by numerical integration.

2. ANALYTICAL RESULTS

Assuming harmonic response \( x_0 = A \cos(\omega t + \phi) \) and substituting this in eqn. (1) and by balancing the first harmonic we get,

\[ A^2 = \frac{F^2}{c^2 \omega^2 + \left( \omega^2 - \alpha - \frac{3}{4} \beta A^2 \right)^2} \]  

which clearly shows either one or three real roots for the amplitude \( A \). The stability of this steady-state amplitude(s) can be obtained using the method of slowly varying amplitudes and writing Substituting the above equation in eqn. (1) and carrying out the harmonic balance after neglecting \( \dot{A} \) and \( \dot{\phi} \), we get a two-dimensional flow:

\[ \dot{A} = g(A, \phi) \text{ and } \dot{\phi} = h(A, \phi) \]

The equilibrium solutions of the above-mentioned flow reflect the steady-state solution given by eqn. (2). Standard linearized stability analysis then suggests that when one real root of \( A \) exists, it is stable. On the other hand, when three real roots exist, then the intermediate value of \( A \) is unstable and the other two are stable. A sudden jump in the frequency response is observed at the points of vertical tangencies, when the excitation frequency is swept with increasing and decreasing values. Consequently a hysteresis phenomenon is demonstrated as the excitation frequency is increased and then decreased. In the amplitude-frequency plane, the loci of the points of vertical tangencies with changing value of the excitation amplitude delineate the primary unstable region.

Higher order unstable regions can be determined by considering a small disturbance \( \eta \) to the steady-state
response $x_0$ and substituting $x = x_0 + \eta$ in eqn. (1) and linearizing in $\eta$, one finally gets the following equation

$$\ddot{\eta} + c\dot{\eta} + \left[\alpha + \frac{3}{2} \beta A^2 + \frac{3}{2} \beta A^2 \cos 2(\omega t + \phi)\right] \eta = 0$$

(3)

The stability of the solution of $\eta = 0$ can be judged using Floquet theory. The primary and higher order unstable regions are bounded by periodic solutions of period $\pi / \omega$ and $2 \pi / \omega$ can be mapped on to the A-$\omega$ plane [7].

Besides the harmonic solution, superharmonic contributions can also be obtained. It should be noted that the presence of odd order superharmonics maintains the symmetry of the Duffing’s equation $x \rightarrow -x$ as $t \rightarrow t + \pi / \omega$.

However, the appearance of even order superharmonics breaks this symmetry and $x(t) \neq -x(t + \pi / \omega)$ which results in dual solution. The generation of even order superharmonics can be followed by considering the loss of stability of solution containing odd order superharmonics.

Around $\omega \approx 3\sqrt{\alpha}$, strong one-third subharmonic solution is generated and the response contains a term like $A_{1/3} \cos \left(\frac{\omega t}{3} + \phi_{1/3}\right)$. The invariance of the equation under the change of variable $t \rightarrow t \pm 2\pi / \omega$ suggests two other possible one-third subharmonics $A_{1/3} \cos \left(\frac{\omega t}{3} + \phi_{1/3} \pm \frac{2\pi}{3}\right)$.

3. EXPERIMENTAL AND NUMERICAL RESULTS

A torsional system, which can be modeled as a hard, Duffing oscillator, was designed. This system is shown in Fig. 1. The torsional stiffness characteristic of this oscillator was obtained by measuring the applied torque ($T_1$) and the resulting twist ($\theta$), which is shown in Fig. 2. The data fits the equation

$$T_s = k_1 \theta + k_2 \theta^3$$

(4)

with $k_1 = 60.14 \text{ N.mm} / \text{rad}$ and $k_2 = 62.46 \text{ N.mm} / \text{rad}^3$. The plot of the free vibration decay is shown in Fig. 3, which suggests a linear viscous damping with a damping factor in the range of 0.014 to 0.016. Harmonic excitation is provided to the system by feeding the coil from a harmonic signal generator via a power amplifier. The response is measured by an accelerometer attached to the oscillating bar magnet. The accelerometer output is sent to an FFT analyzer via a charge amplifier with integrating circuits. The non-dimensional form of the equation of motion for the torsional forced vibration of the system is finally obtained as [8]:

$$\Theta'' + 0.03\Theta' + \Theta + \Theta^3 = T_1 \cos(\omega_1 \tau)$$

(5)

where

$$\Theta = \sqrt{\frac{\beta}{\alpha}}, \quad T_1 = \sqrt{\frac{\beta}{\alpha}}, \quad \omega_1 = \frac{\omega}{\sqrt{\alpha}}, \quad \tau = \sqrt{\alpha} t,$$

$$\alpha = k_1 / J, \quad \beta = k_2 / J$$

and $T = \Gamma / J$ with $\Gamma$ = amplitude of exciting torque, $J$ = moment of inertia of the bar about the axis of rotation (= 4.69x10$^{-3}$ kg m$^2$), $\tau$ denoting the time and the prime denoting differentiation with respect $\tau$.

Equation (5) is numerically integrated with various initial conditions and the results are presented below for various values of the excitation levels (in terms of $\Gamma$ in N.m) and frequencies (in Hz). Interesting features revealed in the frequency response curves, either by numerical integration or by the experiment, are discussed.

Figure 4 shows a typical numerical result obtained by numerical integration. Three different frequency regions are identified and marked as $J_1$, $J_2$ and $J_3$. We consider one frequency in each of these regions, namely 1.8 Hz, 6.6 Hz and 18 Hz, for further detailed study. The irregular behaviour in the region $J_1$ is also verified experimentally, especially as the excitation level is increased. This is shown in Fig. 5.

Numerical simulation at 1.8 Hz with different levels of excitation and various initial conditions exhibits, symmetry breaking, period 2, period 3, period 7 and chaotic responses. These can be clearly seen in Fig. 6. Figure 7 shows experimentally obtained response and power spectrum clearly demonstrating chaotic response. At 6.6 Hz, no chaotic response was obtained, but jump phenomenon is observed even at low levels of excitation, which is also confirmed by the experimental result (Fig. 8).

Around 18 Hz, when the one-third subharmonic appears, if the excitation level is high enough so that the jump from resonance to non-resonance branch also occurs around this frequency, then numerical results indicate sensitivity to small change in frequency. The response can jump unpredictably between three periodic attractors having different periods (Fig. 9). This is what is known as Wada basins of attraction. In this situation, the transients are found to be long lasting as shown in Fig. 10. The experimentally obtained response and corresponding power spectrum at 18 Hz is shown in Fig. 11. This figure clearly shows the effects of Wada basins of attraction, which manifest in unsteady nature of the response.

CONCLUSIONS

1. Jump phenomenon in the response of a hard, Duffing oscillator is revealed by the experimental result.

2. Complicated behaviour of the frequency response curve in the low frequency regime exhibited by both numerical and experimental results can be explained by complex interaction of higher order
unstable regions, especially at high levels of excitation.
3. Chaotic response at low frequency is obtained experimentally.
4. Wada basins of attraction of three simultaneously present periodic solutions are revealed for a hard, Duffing oscillator. This is shown by both numerical and experimental results.

REFERENCES
Fig. 5: Low Frequency Response (Expt.)

Fig. 6: Different Solutions at 1.8 Hz

Fig. 7: Experimental Time Series and Power Spectrum at 1.9 Hz

Fig. 8: Jump Phenomenon (Expt.)
Fig. 9: Three Coexisting Periodic Attractors

Fig. 10: Long Transients (Numerical)

Fig. 11: Time Series and Power Spectrum (Expt.)