Siegel Discs in Complex Dynamics

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1 Introduction and Definitions

A dynamical system is a physical setting together with rules for how the setting changes or evolves from one moment of time to the next or from one stage to the next. A basic goal of the mathematical theory of dynamical systems is to determine or characterize the long term behaviour of the system. The simplest model of a dynamical process supposes that $(n+1)$-th state, z_{n+1} can be determined solely from a knowledge of the previous state z_n, that is $z_{n+1} = f(z_n)$ where f is a function. These systems are often called Discrete Dynamical systems. We shall deal with one such kind of systems namely, *Complex Dynamical Systems*

In the study of *Complex Dynamical Systems*, the evolution of the system is realized by iteration of entire or meromorphic complex functions $f : \mathbb{C} \to \hat{\mathbb{C}}$. For an initially chosen point $z_0 \in \mathbb{C}$, the long term behavior of iterates $\{f^n(z_0)\}$ is of primary importance. For being more precise the following definition is required.

Definition 1.1. *The family $\mathfrak{T}$ of functions defined on the plane is said to be normal at $z \in \mathbb{C}$ if every sequence extracted from $\mathfrak{T}$ has a subsequence which converges uniformly either to a bounded function or to ∞ on each compact subset of some neighbourhood of z.*

In the present context, the family $\mathfrak{T}$ is the sequence of iterates $\{f^n\}_{n>0}$.

A function $f : \mathbb{C} \to \hat{\mathbb{C}}$ is said to be rational if it is of the form $\frac{p(z)}{q(z)}$ where both $p(z)$ and $q(z)$ are complex polynomials not having any common factor. The degree of $f(z)$ is defined by $max\{degree\ p(z), degree\ q(z)\}$. Any other function on $\mathbb{C}$ which is not rational is called transcendental.

By a function, it shall be meant to be a rational function of degree more than one or a transcendental function through out the article.

The set where $\{f^n\}$ is normal, is widely known as Fatou set(or stable set) of f, denoted by $\mathcal{F}(f)$. The complement of $\mathcal{F}(f)$ in the extended complex plane is known as Julia set. A detailed description of Fatou set follows.

Fatou Components:

The Fatou set of a function is open by definition. A Fatou component is a maximal connected open subset of $\mathcal{F}(f)$. A component U of $\mathcal{F}(f)$ is n-periodic if n is the smallest natural number to satisfy $f^n(U) \subseteq U$. Then $\{U, f(U), f^2(U)...f^{n-1}(U)\}$ is called an n-periodic cycle. If $n = 1$, U is called an invariant component. A Fatou component U is said to be completely invariant if it is invariant and satisfies $f^{-1}(U) \subseteq U$. A component U of $\mathcal{F}(f)$ is said to be pre-periodic if there exists a natural number $k > 1$ such that $f^k(U)$ is periodic.

For a given function f, certain points in $\mathbb{C}$ are very crucial in studying the components of $\mathcal{F}(f)$. These are singular values and periodic points of f.

Definition 1.2. *Singular Values: A point w is a critical point of f if $f'(w) = 0$. The value of the function at w, $z = f(w)$ is called critical value of f. A point a is called an asymptotic value of f if there exists a path $\gamma(t)$(a continuous function from $(0,\infty)$ to $\hat{\mathbb{C}}$) satisfying $\lim_{t\to\infty} f(\gamma(t)) = a$. All the critical and asymptotic values of a function are known as singular values. The set of all singular values of a function f is denoted by S_f.*

Definition 1.3. *Periodic point: A point $z \in \mathbb{C}$ is called a p-periodic point of f if p is the smallest natural number satisfying $f^p(z) = z$. If $p = 1$, z is called a fixed point. A periodic point z is said to be attracting, indifferent or repelling if $|(f^p)'(z)| < 1, = 1 or > 1$ respectively. Further, an indifferent periodic point is called irrationally indifferent if t is irrational in the expression $(f^p)'(z) = e^{i2\pi t}$. The periodic point is called rationally indifferent if t is rational.*

A complete classification of periodic Fatou components was made by D.Sullivan [4]for rational functions[1]. His classification holds for transcendental functions with slight modifications.
Suppose U is an n-periodic Fatou component. Then exactly one of the following possibilities occur.

1. *Attracting Basin*: If for all the points z in U, $\lim_{k\to\infty} f^{kn}(z) = p$ where p is an attracting n-periodic point lying in U, the component U is said to be an attracting basin.

2. *Parabolic Basin*: In this case ∂U (the boundary of U) contains a rationally indifferent n-periodic point p. Further $\lim_{k\to\infty} f^{kn}(z) = p$ for all $z \in U$.

3. *Baker Domains*: If for $z \in U$, $\lim_{k\to\infty} f^{kn}(z) = \infty$ then the Fatou component $U$is called a Baker domain.

4. *Herman Rings*: If there exists an analytic homeomorphism $\phi : U \to A$, A is the annulus $\{z : 1 < |z| < r\}, r > 1$, such that $\phi(f^k(\phi^{-1}(z))) = e^{i2\pi\alpha}z$ for some $\alpha \in \mathbb{R}\backslash\mathbb{Q}$, then U is called as Herman ring.

5. *Siegel Disc*: A Fatou component U is said to be a siegel disc if there exists an analytic homeomorphism $\phi : U \to D$ such that $\phi(f^k(\phi^{-1}(z))) = e^{i2\pi\alpha}z$ for some $\alpha \in \mathbb{R}\backslash\mathbb{Q}$. Here D is the unit disc. As definition implies Siegel discs are simply connected.

The existence of Siegel discs were established by C.L.Siegel in 1941 [2, 3]. The bahavior of $\{f^n\}_{n>0}$ on Siegel discs is different in many significant ways than on other kind of components. Some of these aspects are presented in the article. Also non existence of Siegel discs under some condition is dealt with.

The forward orbit of all the singular values of f, defined by $\{f^n(z) : z \in S_f \text{ and } n \in \mathbb{N}\}$ is related to the Siegel discs as stated in the following theorem which appears in [5].

Theorem 1.1. *Suppose f is a function and $C = \{U_0, U_1, U_2 ... U_{p-1}\}$ is periodic cycle of siegel discs or Herman rings, then $U_j \subset \overline{O^+(S_f)}$, the closure of the forward orbit of all the singular values.*

For the sake of simplicity, only invariant Siegel discs are considered in this article though the deliberations holds good for periodic cycle of Siegel discs as well. Through out the article the unit disc $\{z \in \mathbb{C} : |z| < 1\}$ is denoted by D and S stands for an invariant Siegel disc.

2 Results

By definition, $f(z) = \phi^{-1}(\rho(\phi(z))), z \in S$ where $\rho(z) = e^{i2\pi\alpha}z, \alpha \in \mathbb{R} \setminus \mathbb{Q}$. All the functions in the right hand side are one-one, so f is one-one. This fact supports the proofs of various results. The following proposition guarantees the existence of pre-periodic components.

Proposition 2.1. *Suppose the Fatou set of f, $\mathcal{F}(f)$ contains a Siegel disc. Then there are pre-periodic components in $\mathcal{F}(f)$.*

Proof. Given any point $z \in S$, there are more than one point whose f-image is z. This follows from Picard's theorem for transcendental functions and quite obvious for rational functions of degree more than one. As f is one-one on S, any point $z \in S$ has only one pre-image in S. The other pre-images must lie in Fatou components other than S. These components are different from S. If U is such a component, then $f(U) = S$ which is invariant (one-periodic). So U is pre-periodic. □

Remark 2.1. *A Siegel disc S does not satisfy $f^{-1}(S) \subseteq S$. So these are not completely invariant.*

Remark 2.2. *For transcendental functions, $\mathcal{F}(f)$ may contain infinitely many pre-periodic components. This follows from Picard's theorem which states that every point in $\hat{\mathbb{C}}$, except at most two has infinitely many pre-images under f.*

It is clear from the classification of Fatou components that attracting and rationally indifferent periodic points are associated with attracting and parabolic basins respectively. Also it is known that repelling periodic points are in the Julia set[5]. The next proposition finds the association of irrationally indifferent periodic points with Siegel discs.

Proposition 2.2. *An invariant Siegel disc S contains an irrationally indifferent fixed point of f.*

Proof. From definition, it follows that $f(\phi^{-1}(z)) = \phi^{-1}(\rho(z))$ on D where $\phi : S \to D$ is the analytic homeomorphism and $\rho(z)$ is an irrational rotation on D. The origin is fixed by ρ, so $f(\phi^{-1}(0)) = \phi^{-1}(0)$. That means $\phi^{-1}(0)$ is a fixed point of f in S. Being in the Fatou set, this fixed point is either attracting or neutral. Again if it is attracting or rationally indifferent, it must correspond to an attracting basin or a parabolic basin which is not the case. Therefore, the fixed point is irrationally indifferent. □

The previous proposition says that there is a point $z^* \in S$ that remains fixed by f. In other words, $f(z^*) = z^*$. There are also other invariant subsets of S. Precisely, S is a disjoint union of all such invariant subsets. This is the content of the next theorem.

Suppose C_s is a circle of radius s centered at origin where $0 \leq s < 1$. Denote C_s^* by $\phi^{-1}(C_s)$ where ϕ is the analytic homeomorphism that exists from S onto the unit disc D by definition of Siegel discs. Here C_0 and C_0^* are assumed to be 0 and the irrationally indifferent fixed point in S respectively.

Theorem 2.1. *If S is an invariant Siegel disc of f, then $S = \bigcup_{0 \leq s < 1} C_s^*$ where each C_s^* is invariant and $C_s^* \bigcap C_t^* = \emptyset$ for $s, t \in [0\ ,\ 1)$ and $s \neq t$.*

Proof. For any $s \in [0\ ,\ 1)$, $C_s^* \subset S$. So $\bigcup_{0 \leq s < 1} C_s^* \subseteq S$. Let $w \in S$, then $|\phi(w)| < 1$ by definition of ϕ. Denote the circle having radius $|\phi(w)|$ and centered at origin by $C_{|\phi(w)|}$. Now $\phi^{-1}(C_{|\phi(w)|})$ is nothing but $C^*_{|\phi(w)|}$ which contains w. This implies $S \subseteq \bigcup_{0 \leq s < 1} C_s^*$ and $S = \bigcup_{0 \leq s < 1} C_s^*$ follows.
To show that each C_s^* is invariant, let $z \in C_s^* = \phi^{-1}(C_s)$. From the definition of Siegel disc, it follows that $f = \phi^{-1}\rho\phi$ on S. Now, $\phi(z) \in C_s$ and C_s is preserved by ρ. So $f(z) \in C_s^*$. Therefore, C_s^* is invariant.

We get $C_s^* \bigcap C_t^* = \emptyset$ as a consequence of $C_s \bigcap C_t = \emptyset$ and one-one ness of ϕ. □

Corollary 2.1. *All the limit functions of $\{f^n\}$ on S are non constant.*

Proof. Suppose there is a subsequence $\{f_{n_k}\}$ of $\{f^n\}$ that converges uniformly on each compact subset of S to a constant c. Let C_s^* is an invariant curve in S (as in the previous theorem) such that $c \notin C_s^*$. Now for all n_k and $z \in C_s^*$, $f_{n_k}(z) \in C_s^*$. Thus a neighbourhood around c can be found which does not contain any $f_{n_k}(z)$. So $f_{n_k}(z)$ can not converge to c on C_s^* ; a contradiction. Therefore, any limit function of $\{f^n\}$ is non constant. □

This corollary stands in direct contrast with attracting or parabolic Fatou components where all the limit functions of $\{f^n\}_{n>0}$ are constants.

The next result characterize certain functions which do not have Siegel discs in their Fatou set.

Theorem 2.2. *Suppose for a function $f : \mathbb{C} \to \hat{\mathbb{C}}$, the forward orbit of singular values $O^+(S_f) \subseteq \gamma$ where γ is bounded but not a closed curve in $\mathbb{C}$. Then $\mathfrak{F}(f)$ do not contain any Siegel Disc.*

Proof. Suppose S is an invariant Siegel disc of $\mathcal{F}(f)$. By the Theorem 1.1, $\partial S \subset \overline{O^+(S_f)}$. The boundary of S must be a simple closed curve in $\hat{\mathbb{C}}$. But $O^+(S_f)$ is given to be a subset of γ which is bounded. So ∂S must be a simple closed curve in $\mathbb{C}$ which is no longer true as $\partial S \subset \overline{O^+(S_f)} \subseteq \gamma$ and γ is not closed. Thus arises a contradiction implying that no Siegel disc can exist. □

The function $\lambda tanh(e^z)$ for non zero real λ has three singular values namely, $\lambda, -\lambda$ and 0. The forward orbit of all these values is a subset of $\mathbb{R}$ and remains bounded. So no Siegel disc exists in $\mathcal{F}(\lambda tanh(e^z))$. The rational function z^2 is another example where the only singular value 0 is a fixed point, so have bounded forward orbit.

References

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